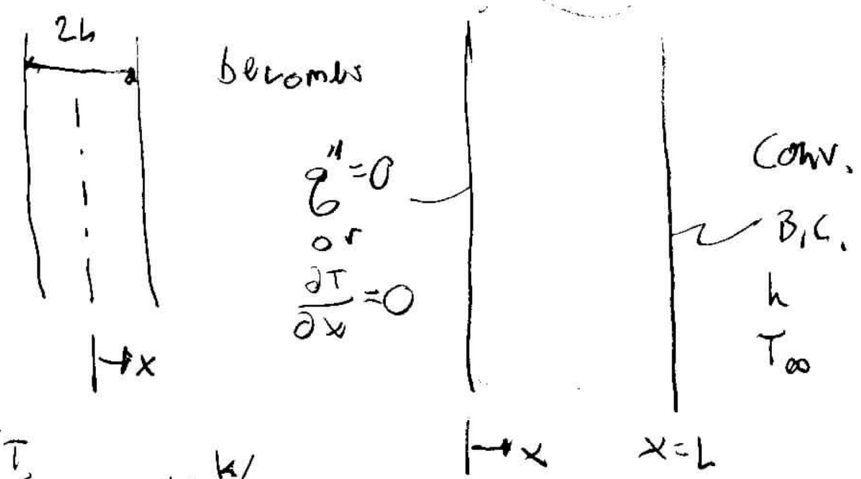
What about finite size objects?

Try cartesian Recall



Cons. of T-energy $\quad \frac{\partial T}{\partial t} = \alpha \frac{\partial^2 T}{\partial x^2} \qquad \alpha = k/\rho c_p$

I.C. $\quad @\ t=0\ ;\ T(0,x) = T_i$

B.C. $\quad @\ x=0\ ;\ \frac{\partial T}{\partial x} = 0$ (insulated)

$\quad @\ x=L\ ;\ -k\left.\frac{\partial T}{\partial x}\right|_{x=L} = h\left[T(L,t) - T_\infty\right]$

Non-dim $\boxed{\mathcal{X} = \frac{x}{L}}$ $\quad \boxed{\theta(t,x) = \frac{T(t,x) - T_\infty}{T_i - T_\infty}}$

will find that

$$\left.\begin{aligned} \frac{\partial T}{\partial t} &= (T_i - T_\infty)\frac{\partial \theta}{\partial t} \\ \frac{\partial^2 T}{\partial x^2} &= \frac{(T_i - T_\infty)}{L^2}\frac{\partial^2 \theta}{\partial \mathcal{X}^2} \end{aligned}\right\} \text{ into } \frac{\partial T}{\partial t} = \alpha \frac{\partial^2 T}{\partial x^2}$$

to get $\quad \frac{\partial \theta}{\partial t} = \left(\frac{\alpha}{L^2}\right)\frac{\partial^2 \theta}{\partial \mathcal{X}^2}$

$$\frac{\partial \theta}{\partial\left(\frac{\alpha t}{L^2}\right)} = \frac{\partial^2 \theta}{\partial \mathcal{X}^2}$$

Now define $\boxed{\tau = \frac{\alpha t}{L^2}}$ (dimensionless!)

Fourier #

$$\boxed{\frac{\partial \theta}{\partial \tau} = \frac{\partial^2 \theta}{\partial \mathcal{X}^2}}$$

Non-dim. the B.C. and initial cond. to get

I.C. $\theta(\tau=0, x)=1$

B.C. $\left.\frac{\partial \theta}{\partial x}\right|_{x=0} = 0 \qquad \left.\frac{\partial \theta}{\partial x}\right|_{x=1} = -\left(\frac{hL}{K}\right)\theta\Big|_{x=1} = -Bi\,\theta\Big|_{x=1}$

Recall $\boxed{Bi = \frac{hL}{K}} \sim \frac{\text{conv. } q \text{ transfer}}{\text{cond. } q \text{ transfer}}$

and $\boxed{\tau = \frac{\alpha t}{L^2}} \sim \frac{t}{(L^2/\alpha)} \sim \frac{t}{t_{char}} \qquad \boxed{t_{char} = L^2/\alpha}$

for cond./conv./diff. problems

Physics of problem

$\frac{\partial \theta}{\partial \tau} = \frac{\partial^2 \theta}{\partial x^2}$ temporal change (rate) of T-energy $\sim$ net cond. to location from left & right (net difusion of T-energy)

$\boxed{\text{B.C. @ } x=0}$

Insulated wall

$\boxed{\text{B.C. } x=1}$

q cond to surface from inside the solid $\sim$ q conv away (or to) the surface by surrounding fluid

Apparently $T(t, x; L, k, \rho, c_p, h, T_i, T_\infty)$

now $\boxed{\theta(\tau, x; Bi)}$

To solve try seperation of variables

$$\Theta(\Upsilon, x) = \Xi(x)\, T(\Upsilon) \qquad \text{j.t.i}$$

and it breaks up our problem into 2 ODEs

$\underbrace{\text{1 in } \Upsilon}_{\text{leads to exponentials in } \Upsilon}$ and $\underbrace{\text{1 in } x}_{\text{leads to } \sin, \cos(x) \text{ with eigenvalues}}$

Work out the details ... ~ to satisfy the B.C. will need a series soln (~ taylor series on steroids)

~ take Calc 4, PDE course, or lots more heat transfer

Bottom line ...

e-values

B.C. @ $x=0$ says you only have $\cos(\lambda x)$ terms

@ $x=1$ says your λ must satisfy $\lambda \tan\lambda = Bi$ charq. eqn.

I.C. @ $\Upsilon = 0$ will determine the constant coef. in your Fourier series solution

Since $\tan(\lambda)$ is periodic, $\lambda \tan\lambda = Bi$ has lots of roots. One is between $[0, \pi]$

" " " $[\pi, 2\pi]$

and so on. So λ_n is between $[(n-1)\pi, \pi]$

Putting it altogether,
$$\theta = X(x)\,T(\tau)$$
$$= A e^{-\lambda^2 \tau} \cos(\lambda x)$$

The linear combination is then

$$\boxed{\theta(\tau, x) = \sum_{n=1}^{\infty} A_n e^{-\lambda_n^2 \tau} \cos(\lambda_n x)}$$

To get the constants A_n, we use our I.C.

$$\theta(\tau = 0, x) = 1$$

or
$$1 = \sum_{n=1}^{\infty} A_n \cos(\lambda_n x)$$

Take advantage of "orthogonality" of e-functions to find that

$$\boxed{A_n = \frac{4 \sin(\lambda_n)}{2\lambda_n + \sin(2\lambda_n)}}$$

We have an analytic, but it is nasty.

However, recall T.S. problems like how many terms to use to be within 1 or 2% of the correct value?

Same thing here... the A_n die off <u>very</u> quickly.

If $\tau > 0.2$, you only need the <u>$n=1$ term</u>!

So $\boxed{\Theta(\tau, x) = \dfrac{T(t,x) - T_\infty}{T_i - T_\infty} \simeq A_1 e^{-\lambda_1^2 \tau} \cos\left(\lambda_1 \dfrac{x}{L}\right)}$ for $\tau > 0.2$

to within 2% of exact value.

(Carry first 2 terms if you wish.)

Recall that $\boxed{A_1(Bi)}$ and $\boxed{\lambda_1(Bi)}$ only. (See Tables for these)

Hey, note that $\Theta(\tau, x=0) = A_1 e^{-\lambda_1^2 \tau} \cdot 1$

or $\boxed{\Theta_{\mathcal{C}} = \Theta_0 = A_1 e^{-\lambda_1^2 \tau}}$

in which case

$$\frac{\Theta(\tau, x)}{\Theta_{\mathcal{C}}(\tau)} = \cos\left(\lambda_1 \frac{x}{L}\right) \quad \text{for } \tau > 0.2$$

so if $\Theta_{\mathcal{C}}$ drops by 20%, so does Θ anywhere else.

Question: How much heat is transfered out of the body when t is some particular time,

$$Q(t) = \iiint_{\forall} \rho c_p \left(T(t,x) - T_i\right) d\forall \quad dx\,dy\,dz$$

so if $t \to \infty$

we know $Q_{max} = m c_p (T_\infty - T_i) = \rho \forall c_p (T_\infty - T_i)$

So we could write

full soln or 1-term approx.?

$$\frac{Q(t)}{Q_{max}} = \frac{\iiint_{\forall} \rho c_p \left(T(t,x) - T_i\right) d\forall}{\rho c_p \forall (T_\infty - T_i)} = \frac{1}{\forall} \iiint_{\forall} (1 - \Theta)\, d\forall$$

Go one step further (1-term approximation)

$$\frac{Q(t)}{Q_{max}} = 1 - \Theta_0(t) \frac{\sin(\lambda_1)}{\lambda_1}$$

Ex) Boil an egg (spherical)

$r = 2.5$ cm or $D = 5$ cm

$T_i = 5°C$ (uniform)

Drop into boiling H_2O $T_\infty = 95°C$

Assume $h = 1200 \frac{W}{m^2 K}$

Calc: how long until $T_{¢}$?

Fooled you.

We are going to heat a flat metal plate.

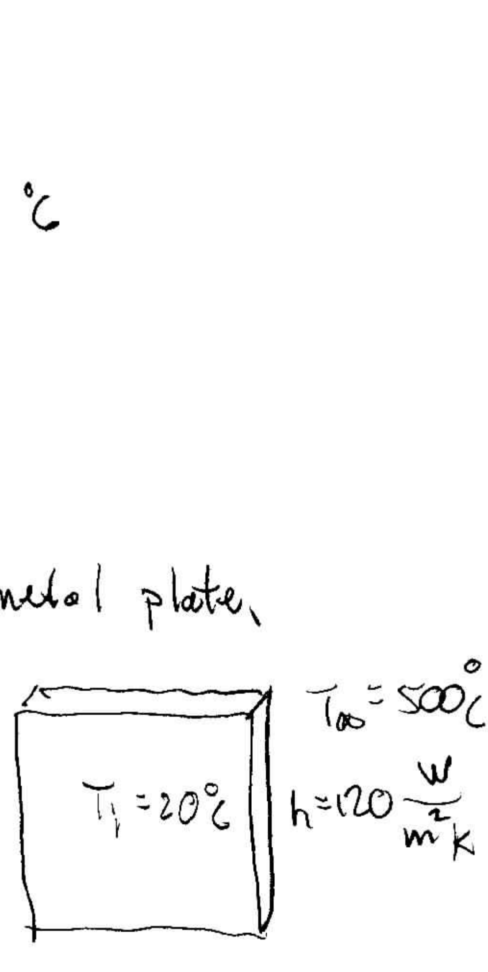
Brass plate $2L = 4$ cm

$T_i = 20°C$

oven temp $T_\infty = 500°C$

Plates are in oven for 7 min

Account for conv. and radiation



combined $h_{conv} = 120\ W/m^2 K$

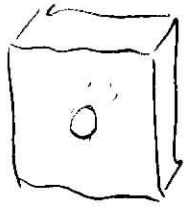
Calc T_{surf} of plate.

Mat'l. properties of brass

$k = 110 \frac{W}{mK}$ $\rho = 8530 \frac{kg}{m^3}$

$C_p = 380 \frac{J}{kg\,K}$ $\alpha = 33.9 \times 10^{-6} \frac{m^2}{s}$

Assume:

-  "∞ large" plate in vert. & horiz.
  analyse small section of thickness $2L = 0.04$ m

- const. material properties

- Fourier\# $\tau > 0.2$ (but will prove it)

Calc $\tau = \frac{\alpha t}{L^2} = \dots = 35.6 > 0.2$ so 1-term is ok

$\frac{1}{Bi} = \frac{k}{hL} = \dots = 45.8$

at surface $x = L$ so $\chi = 1$

$\left.\begin{array}{l} \frac{T_0 - T_\infty}{T_i - T_\infty} = 0.46 \\ \frac{T - T_\infty}{T_0 - T_\infty} = 0.99 \end{array}\right\}$ from Heisler charts

and so (check this out)

$$\frac{T - T_\infty}{T_i - T_\infty} = \underbrace{\left(\frac{T - T_\infty}{T_0 - T_\infty}\right)}_{0.99}\underbrace{\left(\frac{T_0 - T_\infty}{T_i - T_\infty}\right)}_{0.46} = 0.455 \quad \therefore \boxed{T = 282°C}$$

or

from 1-term solution $\dfrac{T(t,x) - T_\infty}{T_i - T_\infty} \approx A_1 e^{-\lambda_1^2 \tau} \cos\left(\lambda_1 \dfrac{x}{L}\right)$

where $A_1(B_i)$ $\lambda_1(B_i)$

But $\begin{cases} Bi = \dfrac{1}{45.8} = 0.02183 \\ \lambda_1 \approx 0.1415 \quad A_1 \approx 1.0033 \end{cases}$ from Table 5.1 p213 in my text

So at surface $x = L$

$$\frac{T(t,x) - T_\infty}{T_i - T_\infty} = A_1 e^{-\lambda_1^2 \tau} \cos(\lambda_1)$$

$T_\infty =$

$T_i =$

$\tau = 35.6$

$$\frac{T - 500}{20 - 500} = 1.0033\, e^{-(0.141)^2 35.6} \cos(0.141)$$

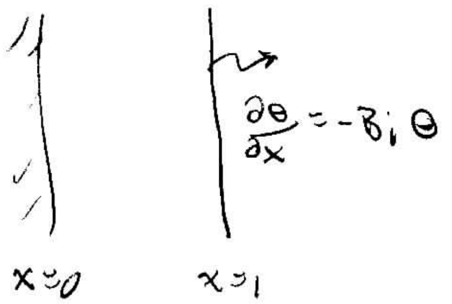
$= 0.489$

so $\boxed{T_{surf} \approx 265\,°C}$

But why not the 282°C ans from charts?

Non-interpolated λ_1 + A_1 values!

Finally, fun with Bi and Convective B.C.



$$\left.\frac{\partial\theta}{\partial x}\right|_{x=1} = -Bi\,\theta\Big|_{x=1}$$

Case (1) if $Bi \to 0$ $\quad \left.\frac{\partial\theta}{\partial x}\right|_{x=1} \approx 0$ $\quad$ Insulated surf. @ $x = 1$

recall $Bi = \frac{hL}{k} \sim \frac{\text{conv}}{\text{cond}}$

so $Bi \to 0$ $\quad \frac{\text{bad conv}}{\text{good cond}}$ $\quad$ also lumped analysis regime

Case (2) $\quad \theta\Big|_{x=1} = -\frac{1}{Bi}\left.\frac{\partial\theta}{\partial x}\right|_{x=1}$

if $Bi \to \infty$ $\quad \theta\Big|_{x=1} \approx 0$ $\quad$ so $\theta = \frac{T - T_\infty}{T_i - T_\infty} \approx 0$

so $Bi \to \infty \sim \frac{\text{good conv}}{\text{bad cond}}$ $\quad$ or $T \approx T_\infty$ fixed surface temp